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ABSTRACT

Engelmann spruce (*Picea engelmannii* Parry ex Engelm.) is a high-elevation species found in northwestern North America. Its importance is increasing as the interest in these high-elevation sites grows. Consequently, growth intercept models that predict site index for this species need to be developed. Previous growth intercept models were fit with nonlinear least squares regression. With this technique, the assumption that the "x" variable (i.e., height) in the fitting is known without error is violated, resulting in a bias. The aim of this study was to remove this source of bias with the errors-in-variable method of moments fitting technique. This involved developing errors-in-variable method of moments estimators for growth intercept models and fitting these models to stem analysis data. Nonlinear least squares regression and the method of moments estimators were then compared to evaluate the significance of the fitting technique. The method of moments models and the regression resulted in almost the same predictions except at ages less than approximately breast height age 15, where the method of moments estimators gave lower site index predictions. The errors-in-variable method of moments should be used to fit growth intercept models because it eliminates a source of bias. Some areas for further research in this technique still exist.

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1 INTRODUCTION

Site index is an important measure of site productivity or site quality (Bontemps and Bouriaud 2014). In British Columbia, site index is defined as the height (m) of a site tree at breast height age (bha) 50 years. Growth intercept models (sometimes also called height intercept models [e.g., Oliver 1972]) are equations that use early height growth of site trees to estimate site index. Growth intercept models are designed specifically to be applied in young stands, particularly for tree species with distinct annual branch whorls (Carmean 1975). Early growth intercept models required a pre-defined number of years growth, (usually 5 years) starting at some specified point above breast height to estimate site index (e.g., Warrack and Fraser 1955; Day et al. 1960; Gregory 1960; Alban 1972, 1979). Growth is obtained by measuring the distance between the internodes demarcating x years of growth and dividing that distance by x.

Engelmann spruce (*Picea engelmannii* Parry ex Engelm.) is a tree species that is found mostly on high-elevation sites (Alexander and Shepperd 1990). In British Columbia, this species has an approximate elevation range of 1200–2100 m in the southwest to 900–1700 m in the north (Meidinger and Pojar 1991). Growth intercept models are particularly needed for Engelmann spruce. Recently, timber harvesting has moved up into high-elevation sites, so the growth intercept method will soon yield Engelmann spruce site index data to aid in making sound forest management decisions for this species.

Engelmann spruce grows predominately in the Engelmann Spruce – Subalpine Fir (ESSF) biogeoclimatic zone, but it can also be found in other biogeoclimatic zones (Meidinger and Pojar 1991; Klinka et al. 2000). It readily hybridizes with white spruce (*Picea glauca* (Moench) Voss) at lower elevations in the interior of British Columbia. For that reason, and because it is difficult to discern between white spruce, Engelmann spruce, and their cross, this study was restricted to the ESSF zone where white spruce and the white × Engelmann cross is rare.

The traditional growth intercept method of estimating site index does not work well for species that do not have distinct annual branch whorls, such as western hemlock (*Tsuga heterophylla* (Raf.) Sarg.). Therefore, the growth intercept method was modified so that it could be applied to tree species without annual branch whorls (Nigh 1996). The modification involves measuring the total height and bha of the site tree above breast height to obtain the growth intercept instead of calculating growth over a fixed number of years. A suite of 50 models, one for each bha (i.e., ring count) from 1 to 50 is then fit to the site index/growth intercept data. To apply the models, the practitioner selects a site tree, measures its height and bha, and calculates the growth intercept. Then, the model associated with the bha of the sample tree is used to estimate site index from the growth intercept. For consistency in field procedures, the modified growth intercept method is now used for all species in British Columbia (British Columbia Ministry of Forests 1995) regardless of whether they have distinct annual branch whorls or not. This method has the advantage of using all the available tree growth to estimate site index, not just growth over a fixed number of years, and results in a more flexible system and more accurate site index estimates.

The growth intercept models in British Columbia have the same functional form for each bha but have different parameter estimates:

$$SI = 1.3 + \alpha_A^* GI^{\beta_A} + \varepsilon \tag{1}$$

$$GI = \frac{h_A - 1.3}{A - 0.5} \tag{2}$$

where $SI = h_{50}$ = site index (m), h_A = total tree height (m) at bha A (years), GI is the growth intercept or mean annual height increment above breast height, α_A^* and β_A are model parameters specific to each bha from 1 to 50, and ε is the random error term with the usual assumption that the ε are identically and independently normally distributed with a mean of 0 and variance of σ^2.

Growth intercept models are available only up to bha 50 because at this age there is little or no model prediction error. Therefore, a seamless transition from the growth intercept method to a site index model can be made at this age because site index models typically also have little or no error at bha 50.

With the modified growth intercept method, stem analysis data from a sample of trees are fit to Equation 1 for bhas 1–50. The fitting is usually done with nonlinear least squares or nonlinear maximum likelihood. It is clear that sample tree height is on both sides of Equation 1. Therefore, this procedure suffers from the same problem that other self-referencing functions, such as site index models, have: the variable h_A is both the dependent and independent variable (Northway 1985; Strub and Cieszewski 2002). This is problematic because in regression, the "x" variable is assumed to be known without error. Not only is it inconsistent to treat h_A as having an error when it is the dependent variable (i.e., the site index) and to being error-free when it is the independent variable, having error in the independent variable results in bias (Seber and Wild 1989). Moreover, with growth intercept models, the errors in the dependent and independent variables are correlated because the same variable (h_A) is on both sides of the equation but is evaluated at different ages (at bha A and bha 50).

The purpose of this research was to derive growth intercept models for Engelmann spruce. To remove bias due to errors in the independent variable, the errors in both the dependent and independent variables and the correlations between them were taken into consideration. This was done by following the derivation of errors-in-variable techniques using the method of moments.

2 MATERIAL AND METHODS

2.1 Data

The data for this study consist of stem analysis data collected on Engelmann spruce. The sampling unit is the site tree. Site trees were selected from a 100-m² plot in accordance with the Site Index – Biogeoclimatic Ecosystem Classification sampling and data standards (British Columbia Ministry of Forests and Range 2009), with the exceptions that there was no upper age limit for the sample trees and they must have been at least 80 years old at

breast height. The sample plots were located throughout most of the 16 subzones in the ESSF biogeoclimatic zone (Meidinger and Pojar 1991). Ninety-two plots were established, but some site trees in the sample were suppressed or damaged and were subsequently rogued out, leaving 84 trees available for analysis.

After establishing a plot and identifying the sample tree, the high side of the tree and breast height (1.3 m above the high side of the tree) were marked on the bark of the tree with marking paint. The tree was then felled and the stem analyzed using the stem splitting and node counting technique (Feng et al. 2006). This technique proceeds by splitting the stem longitudinally to reveal the pith. The measured distance from the point of germination to the pith nodes gives the total height of the tree. Since each pith node represents 1 year of height growth, a set of height–total age pairs was obtained for each tree. According to the standards, height is relative to the high side of the tree, so the height of the high side of the tree above the point of germination was subtracted from the height of each pith node. Breast height age was obtained by subtracting the number of years of growth below breast height from the total age for each height-age pair. Data above bha 50 were disregarded since growth intercept models for bhas greater than 50 are not developed. This analysis-ready data set consisted of height-bha pairs (h_A, A), $A = 1, 2, 3, \ldots,$ 50, where h_A = tree height at bha A. Breast height age A represents $A - 0.5$ years of growth, since the first year of growth is only, on average, half of a year.

2.2 Derivation of the Method of Moments Estimators

The method of moments is a common technique for estimating parameters of errors-in-variable models (Dunn 2004; Gillard and Iles 2009; Gillard 2010). The observed height measurements for this project were highly accurate, so measurement error was negligible, but random variation (likely due mostly to environmental fluctuation) could be significant. This is the main source of error in the dependent and independent variables. The method of moments estimators are derived here for clarity. The covariance matrix from which the estimating equations are derived can also be found in Section 4.2 of Dunn (2004).

The true growth intercept model for bha A, $A = 1, 2, 3, \ldots, 50$, is assumed to be:

$$H_{50} = 1.3 + \alpha_A^* \left(\frac{H_A - 1.3}{A - 0.5} \right)^{\beta_A} \Rightarrow H_{50} - 1.3 = \alpha_A^* \left(\frac{H_A - 1.3}{A - 0.5} \right)^{\beta_A} \tag{3}$$

where H_A is total tree height at bha A and is known without error, and α_A^* and β_A are model parameters, which are unique for each age A. Taking natural logarithms of both sides of Equation 3 gives:

$$\ln(H_{50} - 1.3) = \ln(\alpha_A^*) + \beta_A \ln(H_A - 1.3) - \beta_A \ln(A - 1.3) \tag{4}$$

Due to the above-mentioned errors in height, instead of observing H_A, we observe h_A. Assuming that the error in the observed h_A is multiplicative, the following relationship between h_A and H_A holds: $(h_A - 1.3) = (H_A - 1.3) \times \varepsilon_A^*$, where ε_A^* is a multiplicative error term. Then, using this relationship with $A =$ 50:

$$(h_{50} - 1.3) = (H_{50} - 1.3) \times \varepsilon_{50}^* \Rightarrow \ln(h_{50} - 1.3) = \ln(H_{50} - 1.3) + \ln(\varepsilon_{50}^*)$$

Substituting Equation 4 into this equation for $\ln(H_{50} - 1.3)$ and using the notation $y = \ln(h_{50} - 1.3)$, $\alpha_A = \ln(\alpha_A^*)$, and $\varepsilon_A = \ln(\varepsilon_A^*)$ gives:

$$y = \alpha_A + \beta_A \ln(H_A - 1.3) - \beta_A \ln(A - 0.5) + \varepsilon_{50}$$

Noting that $\ln(H_A - 1.3) = \ln(h_A - 1.3) - \varepsilon_A^*$ and letting $x_A = \ln(h_A - 1.3)$ results in Equation 5:

$$y = \alpha_A - \beta_A \ln(A - 0.5) + \beta_A x_A - \beta_A \varepsilon_A + \varepsilon_{50} \tag{5}$$

Letting $Var(\varepsilon_A) = \sigma_A^2$ and $Var[\ln(H_A - 1.3)] = \sigma^2(A)$ (i.e., the among-tree variance of $\ln(H_A - 1.3)$, which can vary by (A), and assuming that $E[\varepsilon_A] = 0$ and $Cov[\ln(H_A - 1.3), \varepsilon_A] = 0$ for all A, then the equations for the first and second moments are:

$$\begin{aligned} E[x_A] &= E[\ln(h_A - 1.3)] \\ &= E[\ln(H_A - 1.3) + \ln(\varepsilon_A^*)] \\ &= E[\ln(H_A - 1.3)] + E[\varepsilon_A] \\ &= E[\ln(H_A - 1.3)] \end{aligned} \tag{6}$$

and is estimated by $\bar{x}_A$;

$$\begin{aligned} Var(x) &= Var[\ln(h_A - 1.3)] \\ &= Var[\ln(H_A - 1.3) + \ln(\varepsilon_A^*)] \\ &= Var[\ln(H_A - 1.3)] + Var(\varepsilon_A) + 2\,Cov[\ln(H_A - 1.3), \varepsilon_A] \\ &= \sigma^2(A) + \sigma_A^2 \end{aligned} \tag{7}$$

and is estimated by s_{xx};

$$\begin{aligned} E[y] &= E[\alpha_A - \beta_A \ln(A - 0.5) + \beta_A x_A - \beta_A \varepsilon_A + \varepsilon_{50}] \\ &= \alpha_A - \beta_A \ln(A - 0.5) + \beta_A E[x_A] - \beta_A E[\varepsilon_A] + E[\varepsilon_{50}] \\ &= \alpha_A - \beta_A \ln(A - 0.5) + \beta_A E[x_A] \end{aligned} \tag{8}$$

and is estimated by $\bar{y}$;

$$\begin{aligned} Var(y) &= Var\left[\alpha_A + \beta_A \ln\left(\frac{H_A - 1.3}{A - 0.5}\right) + \varepsilon_{50}\right] \\ &= Var[\alpha_A + \beta_A \ln(H_A - 1.3) - \beta_A \ln(A - 0.5) + \varepsilon_{50}] \\ &= Var[\beta_A \ln(H_A - 1.3)] + Var(\varepsilon_{50}) + 2\,Cov[\ln(H_A - 1.3), \varepsilon_{50}] \\ &= \beta_A^2\, Var[\ln(H_A - 1.3)] + \sigma_{50}^2 \\ &= \beta_A^2 \sigma^2(A) + \sigma_{50}^2 \end{aligned} \tag{9}$$

and is estimated by s_{yy};

$$Cov(x, y) = Cov\left[\ln(h_A - 1.3),\ \alpha_A + \beta_A \ln\left(\frac{H_A - 1.3}{A - 0.5}\right) + \varepsilon_{50}\right]$$

$$= \mathrm{Cov}[\ln(H_A - 1.3) + \varepsilon_A, \alpha_A + \beta_A \ln(H_A - 1.3) - \beta_A \ln(A - 0.5) + \varepsilon_{50}]$$
$$= \mathrm{Cov}[\ln(H_A - 1.3) + \varepsilon_A, \beta_A \ln(H_A - 1.3) + \varepsilon_{50}]$$
$$= \beta_A \mathrm{Cov}[\ln(H_A - 1.3), \ln(H_A - 1.3)] + \mathrm{Cov}[\ln(H_A - 1.3), \varepsilon_{50}] + \beta_A \mathrm{Cov}[\ln(H_A - 1.3), \varepsilon_A] + \mathrm{Cov}(\varepsilon_A, \varepsilon_{50})$$
$$= \beta_A \sigma^2(A) + \mathrm{Cov}(\varepsilon_A, \varepsilon_{50})$$
$$= \beta_A \sigma^2(A) + \mathrm{Corr}(\varepsilon_A, \varepsilon_{50}) \times \sqrt{\sigma_A^2 \sigma_{50}^2} \quad (10)$$

and is estimated by s_{xy}. The estimators $\bar{x}_A$, $\bar{y}$, s_{xx}, s_{yy}, and s_{xy} are calculated from the data using standard formulae for means, variances, and covariances.

There are five equations (Equations 6–10) and seven unknowns (α_A, β_A, $E[x_A]$, $\sigma^2(A)$, σ_A^2, σ_{50}^2, and $\mathrm{Corr}(\varepsilon_A, \varepsilon_{50})$). Therefore, the method of moments estimating equations cannot be solved without additional information.

This system of equations is augmented by assuming that the errors in annual growth on the logarithmic scale have a mean of 0 and a constant variance (δ^2) for all A between 1 and 50. These deviations then form a random walk with mean 0, $\mathrm{Var}(\varepsilon_A) = (A - 0.5)\,\delta^2$, and $\mathrm{Corr}(\varepsilon_A, \varepsilon_{50}) = \sqrt{(A - 0.5)/49.5}$ (Cryer and Chan 2008). With this additional assumption, estimators of α_A and β_A can now be obtained from Equations 6–10, as follows:

From the random walk assumption,

$$\sigma_A^2 = (A - 0.5)\,\delta^2, \sigma_{50}^2 = (50 - 0.5)\,\delta^2 \Rightarrow \sigma_A^2 = [(A - 0.5)/49.5]\,\sigma_{50}^2$$

From Equation 7,

$$\sigma^2(A) = s_{xx} - [(A - 0.5)/49.5]\,\sigma_{50}^2$$

From Equation 9,

$$\sigma_{50}^2 = s_{yy} - \beta_A^2 \{s_{xx} - [(A - 0.5)/49.5]\,\sigma_{50}^2\}$$

$$\Rightarrow \sigma_{50}^2 = (s_{yy} - \beta_A^2 s_{xx}) / \{1 - \beta_A^2 [(A - 0.5)/49.5]\}$$

From Equation 10,

$$s_{xy} = \beta_A (s_{xx} - \sigma_A^2) + \mathrm{Corr}(\varepsilon_A, \varepsilon_{50}) \times \sqrt{\sigma_A^2 \sigma_{50}^{\varepsilon}}$$

$$= \beta_A \{s_{xx} - [(A - 0.5)/49.5]\sigma_{50}^2\} + \sqrt{(A - 0.5)/49.5} \times \sqrt{[(A - 0.5)/49.5]\sigma_{50}^2 \sigma_{50}^2}$$

$$= \beta_A s_{xx} - \beta_A [(A - 0.5)/49.5]\,\sigma_{50}^2 + [(A - 0.5)/49.5]\,\sigma_{50}^2$$

$$= \beta_A s_{xx} + [(A - 0.5)/49.5]\,\sigma_{50}^2 (1 - \beta_A)$$

Parameter β_A is estimated from the above equations for s_{xy} and σ_{50}^2, and parameter α_A is estimated with Equation 8. The equation for estimating parameter β_A is a quartic function. A closed form solution for the estimate of parameter β_A exists (Shmakov 2011) but it is much easier to solve numerically using, for example, Solver in Microsoft Excel (Frontline Systems Inc. 2014) or a solver in a statistical package such as SAS (SAS Institute Inc. 2004) or R (R Development Core Team 2011). There are four roots to the equation, so care

must be taken to ensure that the correct root is obtained. Note that estimates of β_A will likely be between 0 and 1. Any root outside this range will likely not be the correct root, but this can be ascertained by plotting the fitted model against the data points.

At bha 50—that is, when A = 50—the above solution is over-specified. In this case, parameter β_{50} can be estimated from Equations 7 and 9, noting that $s_{xx} = s_{yy}$ and $\sigma^2_A = \sigma^2_{50}$. The solution results in $\beta_{50} = 1$, and from Equation 8 and noting that $\bar{x}_A = \bar{y}$, the estimate of α_A is ln(49.5). After back-transforming parameter α_A and substituting these parameters into the growth intercept model (i.e., Equation 1 without the error term), the growth intercept model is SI = h_{50}. This is a pedantic result but may appease readers' curiosity about how this method of estimating parameters of growth intercept models works for this special case.

2.3 Model Fitting

Parameters α_A and β_A were estimated with the estimators developed above and the stem analysis data. Procedure NLIN in SAS (SAS Institute Inc. 2004) was used to find the roots of the estimating equation for β_A. The starting estimate for β_A was arbitrarily set at 0.5, but at bhas close to 50, the starting estimate had to be set closer to 1.0 to get the desired root. For comparison purposes, the growth intercept model (Equation 1) was fitted using nonlinear least squares regression with procedure NLIN in SAS (SAS Institute Inc. 2004). This additional analysis assumes that h_A does not have any error but h_{50} does have error.

Given the complicated formula for β_A, it is not practical to use the delta method to estimate the standard error of β_A and hence the standard error of α_A. Instead, a bootstrap was employed to obtain the variance of both α_A and β_A. This bootstrap method was borrowed from the method described in Sahinler and Topuz (2007) for estimating standard errors for regression parameters. Since the independent variables are random, the observations rather than the errors in the regression are resampled (Sahinler and Topuz 2007). Briefly, this method takes a random bootstrap sample of size *n* with replacement from the observations (where *n* is the number of observations in the original sample), estimates the parameters with the method of moments described above, and repeats this r times. For this project, r = 10 000. The standard deviation of these r parameter estimates is calculated to get the bootstrap standard error of the parameter estimates for α_A and β_A. Since the histograms of the bootstrapped estimates of α_A and β_A are normal shaped, the standard error can be used to construct a confidence interval for the parameter estimates using standard-normal theory (Efron and Tibshirani 1993). To get a confidence interval for α^*_A (the back-transformation of α_A), the endpoints of the confidence interval for α_A are simply back-transformed (Efron and Tibshirani 1993). Finally, it needs to be noted that sometimes the bootstrap method may not work well with small samples sizes (Sahinler and Topuz 2007).

3 RESULTS

Parameters α_A and β_A as estimated by the method of moments are presented in Table 1 for bhas 1–50. Also in Table 1 are α_A^* (from the back-transformation of α_A) and parameters α_A^* and β_A as estimated by nonlinear least squares regression (under the assumption that the independent variables are known without error). Parameters α_A^* and β_A are also plotted in Figure 1 for both the method of moments and the nonlinear least squares estimators. The parameter estimates in this figure are joined by lines to clearly show the trends; however, the parameter estimates are discrete.

The data and fitted models for bhas 5, 15, 25, 35, and 45 are plotted in Figure 2. In these plots, the x variable is the growth intercept as calculated with Equation 2. This gives all plots a common axis for the x and y variables. The fitted models with both the method of moments parameter estimates and the nonlinear least squares regression estimates are shown for comparison.

4 DISCUSSION

Growth intercept models predict height at bha 50 (i.e., site index) from height at some younger bha. Since height is on both sides of the model equation, it is clear that both the dependent and independent variables have error associated with them. The Engelmann spruce growth intercept model derived by the method of moments takes into account the error in the dependent and independent variables. This is in contrast to other methods of estimating model parameters, such as least squares, which assumes that the independent variables are known without error.

The result of this work is a suite of 50 simple models, each with two parameters. These models estimate Engelmann spruce site index from bhas 1–50. Although it may seem onerous to manage 50 models, they can be easily programmed in customized software or in a spreadsheet, which makes the number of equations inconsequential to practitioners.

Table 1 and Figure 1 show that the parameters estimated from the errors-in-variable method of moments and nonlinear least square regression are quite similar, especially at bhas greater than 45. Figure 2 is perhaps more revealing as to the differences in the site index predictions from the two sets of parameters. The difference in site index estimates between the two sets of models is not always trivial and can be up to approximately 1 m, particularly for young and fast-growing trees (i.e., those with a large growth intercept). However, by bha 15, the large differences have diminished and the site index estimates from the two modelling approaches are virtually the same. Since growth intercept models are often applied in very young stands to make silviculture investment decisions, it is prudent to fit the models with the errors-in-variable method of moments approach to reduce the bias associated with ignoring the error in the independent variable.

One area associated with this research that may require more work is the assumption about the errors in growth on the logarithmic scale. These errors were assumed to have a mean of 0 and a variance that is constant from bha 1

TABLE 1 *Growth intercept model parameters and their standard errors in parentheses fit using the method of moments and nonlinear least squares regression for breast height ages 1–50. The standard errors from the method of moments were obtained by bootstrapping. Since α^*_A for the method of moments is a back-transformation of α_A ($\alpha^*_A = e^{\alpha_A}$), no standard errors are reported for α^*_A.*

Breast height age (yr)	Parameter estimates					Breast height age (yr)	Parameter estimates				
	Method of moments			Nonlinear regression			Method of moments			Nonlinear regression	
	α_A	α^*_A	β_A	α^*_A	β_A		α_A	α^*_A	β_A	α^*_A	β_A
1	2.7105	15.0367	0.1597	15.4514	0.1456	26	3.6764	39.5050	0.7843	39.2677	0.7737
	(0.079)		(0.039)	(1.267)	(0.041)		(0.055)		(0.037)	(1.831)	(0.032)
2	3.1311	22.9003	0.3805	23.2661	0.3651	27	3.6769	39.5257	0.7891	39.1334	0.7760
	(0.123)		(0.065)	(2.603)	(0.061)		(0.054)		(0.036)	(1.769)	(0.032)
3	3.2370	25.4585	0.4283	26.8189	0.4333	28	3.6715	39.3090	0.7917	38.9406	0.7792
	(0.138)		(0.072)	(3.293)	(0.066)		(0.052)		(0.035)	(1.689)	(0.031)
4	3.3000	27.1115	0.4582	28.6648	0.4662	29	3.6746	39.4347	0.7980	38.9084	0.7831
	(0.134)		(0.069)	(3.515)	(0.066)		(0.049)		(0.033)	(1.600)	(0.029)
5	3.4087	30.2259	0.5148	31.8432	0.5234	30	3.6806	39.6710	0.8055	39.0484	0.7893
	(0.134)		(0.070)	(3.728)	(0.064)		(0.048)		(0.033)	(1.539)	(0.028)
6	3.4566	31.7092	0.5458	32.7645	0.5455	31	3.6798	39.6369	0.8079	39.0086	0.7917
	(0.129)		(0.068)	(3.665)	(0.062)		(0.047)		(0.032)	(1.497)	(0.028)
7	3.5118	33.5095	0.5802	34.3380	0.5768	32	3.6777	39.5534	0.8093	39.1448	0.7970
	(0.120)		(0.064)	(3.697)	(0.061)		(0.047)		(0.032)	(1.450)	(0.027)
8	3.5627	35.2571	0.6081	35.8993	0.6029	33	3.6867	39.9131	0.8173	39.6020	0.8072
	(0.112)		(0.060)	(3.594)	(0.057)		(0.044)		(0.031)	(1.369)	(0.025)
9	3.5568	35.0516	0.6107	36.3530	0.6164	34	3.6934	40.1806	0.8245	39.8222	0.8141
	(0.106)		(0.058)	(3.422)	(0.054)		(0.042)		(0.030)	(1.297)	(0.024)
10	3.5563	35.0350	0.6172	36.1986	0.6211	35	3.7034	40.5841	0.8343	40.0827	0.8219
	(0.106)		(0.059)	(3.264)	(0.053)		(0.040)		(0.028)	(1.215)	(0.022)
11	3.5782	35.8094	0.6353	36.7619	0.6366	36	3.7217	41.3329	0.8501	40.6180	0.8347
	(0.102)		(0.058)	(3.099)	(0.050)		(0.037)		(0.026)	(1.161)	(0.021)
12	3.5684	35.4614	0.6358	36.7264	0.6428	37	3.7405	42.1175	0.8658	41.3080	0.8494
	(0.101)		(0.058)	(3.060)	(0.050)		(0.034)		(0.024)	(1.096)	(0.020)
13	3.6296	37.6992	0.6777	37.7439	0.6659	38	3.7605	42.9714	0.8811	42.1981	0.8660
	(0.093)		(0.053)	(3.053)	(0.049)		(0.032)		(0.023)	(1.048)	(0.018)
14	3.6381	38.0211	0.6900	37.3535	0.6674	39	3.7844	44.0075	0.8998	43.0823	0.8831
	(0.086)		(0.050)	(2.895)	(0.048)		(0.027)		(0.020)	(0.964)	(0.017)

Breast height age (yr)	Parameter estimates: Method of moments α_A	α^*_A	β_A	Nonlinear regression α^*_A	β_A	Breast height age (yr)	Parameter estimates: Method of moments α_A	α^*_A	β_A	Nonlinear regression α^*_A	β_A
15	3.6440	38.2442	0.7024	37.2845	0.6756	40	3.8099	45.1454	0.9185	44.1451	0.9015
	(0.081)		(0.047)	(2.713)	(0.045)		(0.024)		(0.017)	(0.905)	(0.015)
16	3.6461	38.3263	0.7109	37.3762	0.6847	41	3.8197	45.5908	0.9275	44.6917	0.9123
	(0.077)		(0.046)	(2.597)	(0.044)		(0.022)		(0.016)	(0.856)	(0.014)
17	3.6415	38.1493	0.7155	37.4958	0.6941	42	3.8305	46.0850	0.9367	45.5344	0.9273
	(0.074)		(0.045)	(2.519)	(0.043)		(0.020)		(0.015)	(0.818)	(0.013)
18	3.6506	38.4994	0.7260	37.6012	0.7009	43	3.8372	46.3976	0.9435	46.1668	0.9392
	(0.073)		(0.044)	(2.447)	(0.042)		(0.019)		(0.014)	(0.766)	(0.012)
19	3.6597	38.8501	0.7368	38.0907	0.7144	44	3.8459	46.8023	0.9517	46.5971	0.9480
	(0.073)		(0.045)	(2.394)	(0.041)		(0.018)		(0.013)	(0.685)	(0.011)
20	3.6577	38.7709	0.7419	38.3126	0.7248	45	3.8572	47.3316	0.9612	47.1103	0.9575
	(0.070)		(0.044)	(2.307)	(0.040)		(0.015)		(0.011)	(0.602)	(0.010)
21	3.6517	38.5404	0.7452	38.1141	0.7289	46	3.8656	47.7332	0.9693	47.6667	0.9680
	(0.066)		(0.042)	(2.215)	(0.039)		(0.012)		(0.009)	(0.505)	(0.008)
22	3.6580	38.7846	0.7532	38.3663	0.7374	47	3.8757	48.2180	0.9774	48.2097	0.9771
	(0.067)		(0.043)	(2.135)	(0.037)		(0.009)		(0.007)	(0.385)	(0.006)
23	3.6606	38.8850	0.7587	38.6035	0.7455	48	3.8872	48.7765	0.9864	48.7285	0.9856
	(0.066)		(0.043)	(2.081)	(0.036)		(0.006)		(0.004)	(0.279)	(0.004)
24	3.6659	39.0912	0.7665	39.0047	0.7571	49	3.8964	49.2254	0.9941	49.1523	0.9931
	(0.063)		(0.041)	(2.036)	(0.036)		(0.003)		(0.002)	(0.165)	(0.003)
25	3.6696	39.2344	0.7743	39.0885	0.7646	50	3.9020	49.5000	1.0000	49.5000	1.0000
	(0.059)		(0.039)	(1.928)	(0.034)		—[a]		—[a]	—[a]	—[a]

a Standard errors for the parameters for breast height age 50 do not exist because at this age the site index is, by definition, equal to the height, so no model is required to estimate site index.

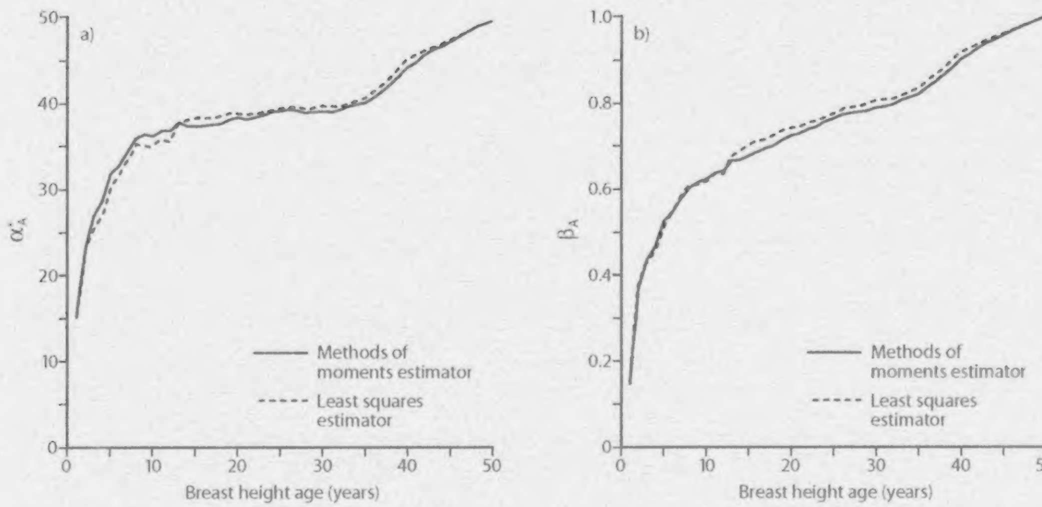

FIGURE 1 *Estimates of parameter α^*_A (a) and β_A (b) versus breast height age (bha), showing the trends from bha 1–50.*

to 50. This leads to the errors being a random walk, which in turn makes the solution to the estimating equations tractable and relatively straightforward. The true and observed growth on the logarithmic scale are:

$$\ln(H_{A+1} - 1.3) - \ln(H_A - 1.3) \text{ and}$$
$$\ln(h_{A+1} - 1.3) - \ln(h_A - 1.3),$$

respectively. Consequently, the error in the observed growth is $[\ln(H_{A+1} - 1.3) - \ln(H_A - 1.3)] - [\ln(h_{A+1} - 1.3) - \ln(h_A - 1.3)]$, which is just the difference between the true and observed annual relative height growth rates (see Mencuccini et al. 2007 for the formula for relative height growth rate). It is also easy to show that this error is just $\varepsilon_A - \varepsilon_{A+1}$. Since both ε_A and ε_{A+1} are assumed to have an expected value of 0, it is reasonable to assume that the error in growth (on the logarithmic scale) has a mean of 0, but it is less clear that these errors should have a constant variance.

Once β_A is estimated, an estimate of σ^2_{50}, which is the variance of the error in height at bha 50 on the logarithmic scale, can be recovered from Equation 9. This was done for bhas 1–49, and these estimates were plotted against bha (Figure 3). The estimates of σ^2_{50} declined over bha. Philosophically, σ^2_{50} should be independent of bha and should be constant. However, at the same time, the growth intercept model for each bha is being fit independently of each other, and from this perspective, there is no reason why σ^2_{50} should be the same for the different bhas. It was hypothesized that perhaps the variation in σ^2_{50} was caused by incorrectly specifying the error structure. To test this, a variety of ad hoc variance and correlation structures for ε_A was employed in the fitting of the models, but none came close to addressing this issue. This is another area for potential research.

No assumption about the distribution of ε_A was required in the derivation of the method of moments estimators. However, if it is assumed that the ε_A were normally distributed, then ε^*_A would be log-normally distributed. Perhaps this information could be used to develop other error structures.

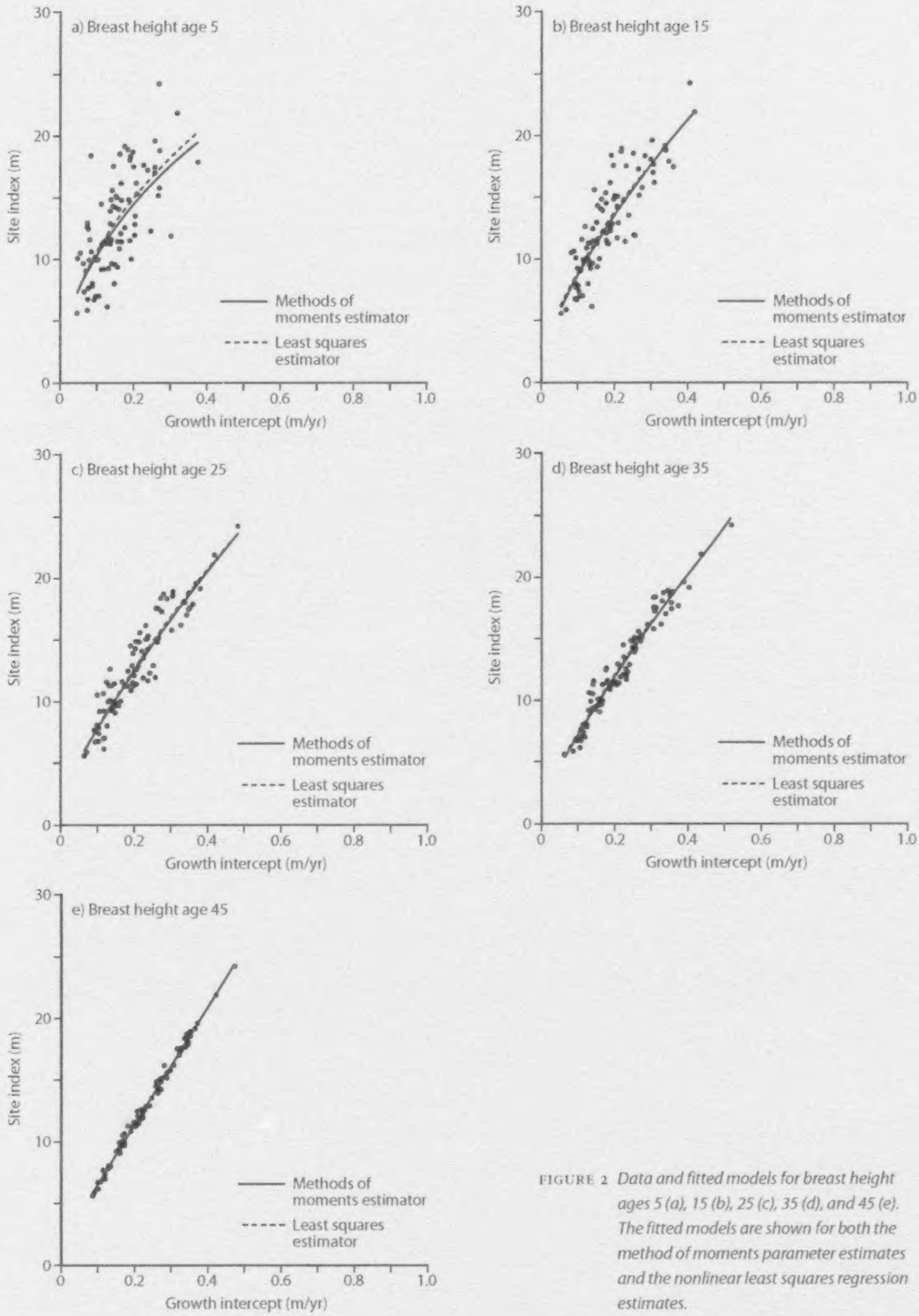

FIGURE 2 *Data and fitted models for breast height ages 5 (a), 15 (b), 25 (c), 35 (d), and 45 (e). The fitted models are shown for both the method of moments parameter estimates and the nonlinear least squares regression estimates.*

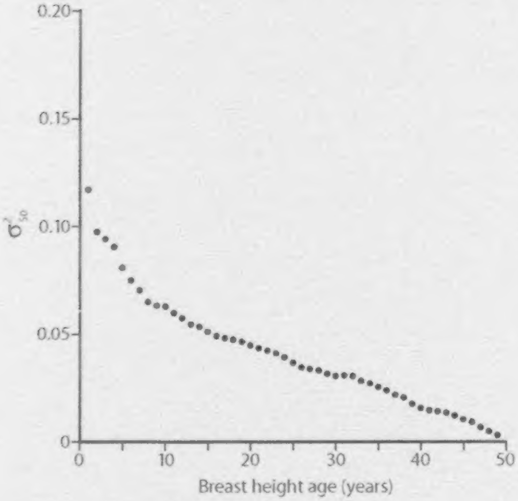

FIGURE 3 *Plot of σ^2_{50} against breast height age.*

5 CONCLUSION

Both the independent and dependent variables in growth intercept models have error, which violates a condition for fitting the models with least squares regression. An errors-in-variable method of moments estimator was used to develop a suite of growth intercept models for Engelmann spruce that accounts for errors in the independent variables. When compared to models developed using traditional nonlinear least squares regression analysis, the site index estimates were nearly identical for bhas greater than 15 but could be up to approximately 1 m different at younger ages. Nevertheless, since the method of moments explicitly accounts for the error in the independent variable, it should be preferred for theoretical reasons if not for practical reasons. Standard errors for the parameter estimates are obtained via bootstrapping.

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